Forward-backward equations for nonlinear propagation in axially invariant optical systems

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We present a general framework to deal with forward and backward components of the electromagnetic field in axially invariant nonlinear optical systems, which include those having any type of linear or nonlinear transverse inhomogeneities. With a minimum amount of approximations, we obtain a system of two first-order equations for forward and backward components, explicitly showing the nonlinear couplings among them. The modal approach used allows for an effective reduction of the dimensionality of the original problem from 3 +1 (three spatial dimensions plus one time dimension) to 1+1 (one spatial dimension plus one frequency dimension). The new equations can be written in a spinor Dirac-like form, out of which conserved quantities can be calculated in an elegant manner. Finally, these equations inherently incorporate spatiotemporal couplings, so that they can be easily particularized to deal with purely temporal or purely spatial effects. Nonlinear forward pulse propagation and nonparaxial evolution of spatial structures are analyzed as examples.

I. INTRODUCTION

Nonlinear propagation of light pulses in dielectric media such as optical fibers has been traditionally modeled using the nonlinear Schrödinger equation (NLSE) [1]. However, it is well known that the NLSE needs modifications to describe a number of higher-order nonlinear effects which become important at increasing powers and for short pulses. Recently, access to new optical systems in which nonlinearities can be considerably enhanced together with the experimental availability of ultrashort pulses has pushed the description based on the NLSE and its modified versions to a limit. A typical example of this scenario is provided by the phenomenon of supercontinuum generation in photonic crystal fibers [2], which requires a specific modeling that goes beyond approaches based on conventional versions of the NLSE [3,4]. These approaches are expressed in evolution equations that differ from the NLSE in the amount of approximations needed to achieve them. We can mention the so-called generalized NLSE [5,6], the nonlinear envelope equation (NEE) [7], the forward Maxwell equation (FME) [4], and the unidirectional pulse propagation equation (UPPE) [8]. Briefly, the aim of these equations is to describe pulse propagation in the regime where the frequency width of the pulse is comparable to the carrier frequency, which, in turn, translates into the fact that the usual approaches such as the slowly varying approximation no longer hold. The specific form of these equations is, on the one hand, the result of applying some other different approximations—e.g., assuming propagation in an homogeneous medium (NEE and UPPE) or single-mode propagation in fibers (generalized NLSE and

FME). On the other hand, two common features of all of them are the neglect of the backward components of the electromagnetic field and their first-order character.

The role played by backward components has been previously analyzed for an homogeneous medium [9]. In this paper we explicitly unveil their role but in the more general case of an axially invariant inhomogenous nonlinear medium by explicitly finding the coupled first-order equations that drive the forward and backward components of the electromagnetic field in an axially invariant nonlinear system. We will show that these first-order forward-backward equations (FBE's) are equivalent to the original second-order equations for the electric components of the electromagnetic field. The FBE's will provide us with a common framework that can encompass different nonlinear phenomena. In fact, since these equations explicitly manifest the couplings between spatial and time-frequency degrees of freedom typical of spatio-temporal phenomena, they can be easily applied to describe either purely temporal or purely spatial effects within the same framework, revealing the total generality of this formalism.

The paper is organized as follows. In Sec. II we derive the most general modal second-order equation for nonlinear propagation in an axially invariant inhomogenous medium and we explain the nature of the only approximation needed to obtain it. In Sec. III we demonstrate the equivalence between the modal second-order equation and the first-order FBE's. In Sec. IV we introduce a spinor representation to obtain a Dirac-like form of the FBE's. In Sec. V we derive the conserved quantities associated with the FBE's and analyze them in the light of phase symmetries. Finally, in Sec.

VI we examine two different nonlinear phenomena occurring in axially invariant inhomogeneous dielectric media in the light of the FBE's: forward pulse propagation and non-paraxial evolution of spatial structures.

II. MODAL SECOND-ORDER EQUATION

The most general equation for the propagation of the electric components of an electromagnetic field in a inhomogeneous, isotropic, and spatially local dielectric medium is given by

$$\nabla^2 \mathbf{E}_\omega - \nabla \left(\nabla \cdot \mathbf{E}_\omega \right) + k_0^2 n_\omega^2 \mathbf{E}_\omega = - \mathbf{G}_\omega^{NL}(\mathbf{E}), \qquad (1)$$

where $\mathbf{E}_\omega = \mathbf{E}_\omega(x,y,z)$ is the complex ω frequency component of the real electric field $\mathbf{E}(x,y,z,t) = \int_\omega \mathbf{E}_\omega(x,y,z) \times \exp(-i\omega t)$, ∇ is the three-dimensional spatial gradient operator, $k_0 = \omega/c$ is the vacuum wave number, $n_\omega = n_\omega(x,y,z)$ is the frequency-dependent refractive index profile of the dielectric medium, and $\mathbf{G}_\omega^{NL}$ is the function that represents the local nonlinear response of the medium to the propagating field. In the most general case, the relative dielectric constant ($\epsilon_\omega = n_\omega^2$) and the nonlinear function $\mathbf{G}_\omega^{NL}$ can be either spatially local (the case we are considering) or nonlocal if there exist some type of spatially delayed response effects. Besides the general wave equation (1), Maxwell's equations require the constraint $\nabla \cdot \mathbf{D}_\omega = 0$ to be satisfied where the displacement field $\mathbf{D}_\omega$ is itself a function of the propagating electric field $\mathbf{E}_\omega$. We assume a spatially local response—that is, $\mathbf{D}_\omega = \epsilon_\omega(\mathbf{E}_\omega)\mathbf{E}_\omega$. In the case the system presents some type of anisotropy, $\epsilon_\omega(\mathbf{E}_\omega)$ is a second-order tensor. Here we consider an isotropic medium, so that $\epsilon_\omega(\mathbf{E}_\omega)$ will be a scalar function, although the generalization to an anisotropic medium is straightforward. The constraint relation has thus the form

$$\nabla \cdot \left[\epsilon_\omega(\mathbf{E}_\omega)\mathbf{E}_\omega \right] = 0. \qquad (2)$$

In most cases in nonlinear optics, the general constraint (2) is replaced by the approximated, and much simpler, "scalar" condition $\nabla \cdot [\epsilon_\omega(\mathbf{E}_\omega)\mathbf{E}_\omega] \approx \epsilon_\omega(\mathbf{E}_\omega)\nabla \cdot \mathbf{E}_\omega = 0$, implying that $\nabla \cdot \mathbf{E}_\omega \approx 0$. This approximation is known to work well in a panoply of nonlinear effects with few and remarkable exceptions, related mainly to extreme self-focussing events [8]. In order to simplify our approach, this approximation will be assumed throughout, although our main results will remain valid in a slightlier general context in which the condition $\nabla \cdot \mathbf{E}_\omega \approx 0$ will not be strictly necessary.

Our interest lies in describing propagation in an axially invariant system—that is, a system in which both the refractive index profile $n_\omega(x,y,z) = n_\omega(x,y)$ as well as all other macroscopic nonlinear structural functions, such as nonlinear susceptibilities, are z independent. In other words, we will focus on systems for which the axial z dependence is carried by the propagating field $\mathbf{E}(x,y,z)$ exclusively. Mathematically, this property is reflected in the fact that there is no explicit dependence on z in the nolinear function $\mathbf{G}^{NL}$. In these systems, the wave equation (1) and the constraint $\nabla \cdot \mathbf{E}_\omega \approx 0$ merge into a single equation:

$$\left(\frac{\partial^2}{\partial z^2} + \nabla_t^2 + k_0^2 n_\omega^2(x,y) \right) \mathbf{E}_\omega = - \mathbf{G}_\omega^{NL}(\mathbf{E}), \qquad (3)$$

where now ∇_t stands for the transverse gradient operator and the axial second-order derivative has been made explicit. Let us now perform a modal expansion of the electric field in terms of the eigenmodes of the linear z-independent system, described by the linear transverse operator $L_0 \equiv \nabla_t^2 + k_0^2 n_\omega^2(x,y)$. These modes fulfill the linear eigenvalue equation

$$L_0 \Phi_n^\omega(x,y) = \beta_n^2(\omega)\Phi_n^\omega(x,y). \qquad (4)$$

Since L_0 does not mix the spatial (polarization) components of the electric field, it is proportional to the identity operator in polarization space. For this reason, its eigenmodes have a three fold degeneracy in polarization indices. Therefore, every multiplet of eigenmodes is constituted by three linearly independent modes, each one proportional to a three-dimensional vector belonging to a basis of $\mathbb{R}^3$. For simplicity, we can consider this basis to be the canonical one $\mathbf{\Phi}_{n(\sigma)}^\omega(x,y) = \Phi_n^\omega(x,y)\mathbf{u}_{(\sigma)}$ ($\sigma = 1,2,3$) where the components of the canonical basis $\{\mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \mathbf{u}_{(3)}\}$ satisfy $\mathbf{u}_{(\sigma)a} = \delta_{\sigma a}$ ($a = 1,2,3$). If the system is lossless, then $n_\omega = n_\omega^*$ and L_0 is self-adjoint; if it is not, we consider the real part of n_ω in Eq. (4) so that L_0 is again self-adjoint. In this way, the set of L_0 eigenmodes $\{\mathbf{\Phi}_{n(\sigma)}^\omega\}$ forms an orthogonal basis of functions of the transverse coordinates that can be used to expand the electric field $\mathbf{E}_\omega(x,y,z)$ at a given z:

$$\mathbf{E}_\omega(x,y,z) = \sum_{n,\sigma} c_{n,\sigma}(\omega;z)\Phi_n^\omega(x,y)\mathbf{u}_{(\sigma)}$$

or, in components,

$$E_\sigma^\omega(x,y,z) = \sum_n c_{n\sigma}(w;z)\Phi_n^\omega(x,y). \qquad (5)$$

Of course, the modal expansion (5) has to be understood as a generalized form of expansion over the entire spectrum of the L_0 operator, in which both discrete and continuum parts of the spectrum can coexist. In practice, however, when dealing with numerical applications, the continuum part is discretized by means of some convenient election of boundary conditions, so that the discretized form of the expansion (5) is, in fact, the one that is used. Since the election of boundary conditions is an artifact of the simulation, one has to make sure that physical results are independent of them.

Now, we substitute the modal expansion (5) into Eq. (3), we multiply this equation by $\Phi_m^{\omega*}$, taking into account that $L_0 \Phi_n^\omega = \beta_n^2 \Phi_n^\omega$, and after integrating the result over the entire transverse space (considering, without loss of generality, that the functions Φ_n^ω are orthonormal: $\int_{\mathbb{R}^2} \Phi_n^{\omega*} \Phi_n^\omega = \delta_{nm}$), we obtain the evolution equation for the modal expansion coefficients c:

$$\left(\frac{\partial^2}{\partial z^2} + \beta_n^2(\omega) \right) c_{n\sigma}(\omega;z) = - F_{n\sigma}^\omega(c). \qquad (6)$$

The equation above represents an effective lower-dimensional equation [$1+1$ dimensions, instead of the $3+1$ dimensions of the original equation (3)] for the coefficients

$c_{n\sigma}(\omega; z)$. Physically, they represent the frequency, modal, and polarization spectral content of the propagating electric field. The nonlinear function $F_{n\sigma}^{\omega}(c)$ is given by

$$F_{n\sigma}^{\omega}(c) = \int_{\mathbb{R}^2} \Phi_n^{\omega*} G_{\omega\sigma}^{NL}\left(\sum c_{n\sigma}\Phi_n^{\omega}\right). \tag{7}$$

This function is an input of the effective equation (6) because it is known once the linear amplitudes are determined out of the linear eigenvalue problem in Eq. (4). The same applies to the propagation constants $\beta_n(\omega)$, which also appear as an input.

As an example of the nonlinear function $F_{n\sigma}^{\omega}(c)$, it is instructive to consider the case of a Kerr nonlinearity, for which $G_{\omega\sigma}^{NL} = \omega^2/c^2 \int_{\omega_2\omega_3} \chi_{\sigma\tau'\sigma'\tau}^{(3)}(\omega,\omega_2,\omega_3,\omega+\omega_2-\omega_3) \times E_{\tau'}^{\omega_2*} E_{\sigma'}^{\omega_3} E_{\tau}^{\omega+\omega_2-\omega_3}$. From the definition of the nonlinear function (7), one finds that F has the form

$$F_{n\sigma}^{\omega}(c) = \sum_{m'n'm} \sum_{\tau'\sigma'\tau}$$
$$\times \int_{\omega_2\omega_3} c_{m'\tau'}^{\omega_2*} T_{\sigma\tau'\sigma'\tau}^{nm'n'm}(\omega,\omega_2,\omega_3,\omega+\omega_2$$
$$-\omega_3) c_{n'\sigma'}^{a_3} c_{m\tau}^{\omega+\omega_2-\omega_3}.$$

Thus, for a Kerr nonlinearity, all the information about the nonlinear properties of the system is encoded in the tensor function T. As a first approximation, one could say that this tensor function depends on frequency both through the susceptibility $\chi^{(3)}$ and through the overlapping integral of linear mode amplitudes:

$$T_{\sigma\tau'\sigma'\tau}^{nm'n'm}(\omega,\omega_2,\omega_3,\omega+\omega_2-\omega_3)$$
$$\approx \omega^2/c^2 \chi_{\sigma\tau'\sigma'\tau}^{(3)}(\omega,\omega_2,\omega_3,\omega+\omega_2-\omega_3)$$
$$\times \int_{\mathbb{R}^2} \Phi_n^{\omega*}\Phi_{m'}^{\omega_2*}\Phi_{n'}^{\omega_3}\Phi_m^{\omega+\omega_2-\omega_3}.$$

Besides, the dependence on modal indices of the T tensor is a result of the overlapping of spatial amplitudes, whereas polarization indices are provided by the third-order susceptibility.

We do not want to get into much detail on the particular form of the nonlinear function since, for our purposes, it is not necessary to provide an explicit form for F in Eq. (6). Only general properties of F will be used and the latter can be inferred from the construction previously described. Along the same line of reasoning, it should be added that the effective equation (6) remains valid in some cases when the "scalar" approximation $\nabla \cdot \mathbf{E}_\omega \approx 0$ no longer holds. Less restrictive approximations can be made in which $\nabla \cdot \mathbf{D}_\omega \approx 0$ but $\nabla \cdot \mathbf{E}_\omega \neq 0$ and, nevertheless, the effective equation (6) retains, formally, its validity. It would differ in the fact that the values of β would be now calculated including the missing vector term in the linear eigenvalue equation (4) and that the nonlinear function F would include also terms of vectorial origin. Again, the analysis of such topics is out of the scope of this paper, for our aim is to transform the almost completely general second-order equation (6) into a first-order formalism in which forward and backward components explicitly appear.

III. DERIVATION OF FORWARD-BACKWARD EQUATIONS

A convenient way of considering the effective equation (6) is as a limit of a equation for a frequency-discretized spectral function $c_{n\sigma}^j(z) \equiv c_{n\sigma}(\omega_j; z)$. This is a typical situation in numerical simulations in which the frequency appears discretized in a Fourier series and one approaches the continuum-frequency limit numerically or likewise in some experimental cases where one physically works with frequency combs instead of continuum sources. In such situations, the frequency discretized version of Eq. (6) is

$$\left(\frac{d^2}{dz^2} + \beta_{nj}^2\right)c_{n\sigma}^j(z) = -F_{n\sigma}^j(c). \tag{8}$$

The general second-order equation (8), displaying all frequency, modal, and polarization indices, will be our starting point. The function $F(c)$ includes all nonlinear contributions in the frequency domain constructed out of Maxwell's equations according to the procedure described in the previous section. As seen above for a Kerr nonlinearity in Maxwell's equation, this function provides a cubic behavior in the spectral function c, which in the discretized frequency version is

$$F_{n\sigma}^j(c) = \sum_{m'n'm}\sum_{\tau'\sigma'\tau}\sum_{k'j'} c_{m'\tau'}^{k'*}\left(T_{nm'n'm\atop\sigma\tau'\sigma'\tau}^{jk'l(j+k'-j')}\right)c_{n'\sigma'}^l c_{m\tau}^{j+k'-j'},$$
$$\tag{9}$$

where T is a tensor in frequency, modal, and polarization indices that can be explicitly calculated out of the amplitudes of the linear fiber modes. However, in order to obtain the first-order counterpart of the general second-order equation (8), it is not necessary to specify a particular form for the nonlinearity function F. It is interesting to stress again that both the modal dispersion relations $\beta_n(\omega)$ (or its discretized-frequency version β_{nj}) and F are inputs that are known once the mode linear problem is solved. In order to simplify the notation, we will incorporate the polarization index into the modal one when dealing with the spectral function c and the nonlinear function F. This is possible because polarization and modal indices always come together in pairs, so that, from now on and without loss of generality, n represents the index pair (n, σ).

We perform now an axial Fourier transform on the spectral functions c in Eq. (8), defined as

$$c_n^j(z) = \frac{1}{2\pi}\int d\beta \tilde{c}_n^j(\beta)e^{i\beta z}$$

or, inversely,

$$\tilde{c}_n^j(\beta) = \int dz c_n^j(z)e^{-i\beta z}.$$

With this definition, $\beta \rightarrow -id/dz$ (or, $d/dz \rightarrow i\beta$), and, thus, the axial Fourier transfom of Eq. (8) is given by

$$(-\beta^2 + \beta_{nj}^2)\tilde{c}_n^j(\beta) = -\tilde{F}_n^j(\tilde{c}), \qquad (10)$$

where $\tilde{F}(\tilde{c})$ is the axial Fourier transform of the nonlinear term in Eq. (8): $\tilde{F} \equiv \mathcal{F}_\beta[F]$. Dividing this equation by $-\beta^2 + \beta_{nj}^2$ (we assume some kind of "$i\epsilon$" prescription to deal with the poles of the inverse of this function in the β plane), we obtain

$$\tilde{c}_n^j(\beta) = \frac{1}{\beta^2 - \beta_{nj}^2}\tilde{F}_n^j(\tilde{c}). \qquad (11)$$

The function preceding the nonlinear factor in the above equation is the axial Green function in β space and has single poles at $+\beta_{nj}$ and $-\beta_{nj}$, corresponding to forward and backward propagation, respectively, for the positive-frequency part of the spectral function $\tilde{c}$ (peaked at a positive frequency $+\omega_0$). Clearly, we can decompose the Green function in the contributions corresponding to the two poles by means of the following identity:

$$\frac{1}{(\beta^2 - \beta_{nj}^2)} = \frac{1}{2\beta_{nj}}\left(\frac{1}{\beta - \beta_{nj}} - \frac{1}{\beta + \beta_{nj}}\right), \qquad (12)$$

which allows us to write Eq. (11) as

$$\tilde{c} = \tilde{c}_F + \tilde{c}_B,$$

where

$$(\tilde{c}_F)_n^j = \frac{1}{2\beta_{nj}}\frac{1}{\beta - \beta_{nj}}\tilde{F}_n^j(\tilde{c}),$$

$$(\tilde{c}_B)_n^j = -\frac{1}{2\beta_{nj}}\frac{1}{\beta + \beta_{nj}}\tilde{F}_n^j(\tilde{c})$$

or, equivalently,

$$(\beta - \beta_{nj})(\tilde{c}_F)_n^j = \frac{1}{2\beta_{nj}}\tilde{F}_n^j(\tilde{c})$$

$$(\beta + \beta_{nj})(\tilde{c}_B)_n^j = -\frac{1}{2\beta_{nj}}\tilde{F}_n^j(\tilde{c}).$$

Now, we take the inverse axial Fourier transform of the above equations, taking into account the previous definitions, so that, considering that $\beta \rightarrow -id/dz$, $\mathcal{F}_z^{-1}(\tilde{c}_{F,B}) = \mathcal{F}_z^{-1}(\mathcal{F}_\beta(c_{F,B})) = c_{F,B}$, and $\mathcal{F}_z^{-1}(\tilde{F}) = \mathcal{F}_z^{-1}(\mathcal{F}_\beta(F)) = F$, we can write the following two first-order equations for forward (F) and backward (B) components:

$$\left(-i\frac{d}{dz} - \beta_{nj}\right)(c_F)_n^j = \frac{1}{2\beta_{nj}}F_n^j(c), \qquad (13)$$

$$\left(-i\frac{d}{dz} + \beta_{nj}\right)(c_B)_n^j = -\frac{1}{2\beta_{nj}}F_n^j(c), \qquad (14)$$

where the total spectrum is given by the sum of forward and backward contributions,

$$c = c_F + c_B. \qquad (15)$$

Equations (13) and (14) are the exact first-order equations equivalent to the general second-order equation (8).

Restoring the continuum frequency notation $[c^j \rightarrow c(\omega)]$, we obtain the first-order forward-backward equations in the continuum-frequency limit:

$$\left(-i\frac{\partial}{\partial z} - \beta_n(\omega)\right)c_F^n(\omega, z) = \frac{1}{2\beta_n(\omega)}F_n(c(\omega, z)), \qquad (16)$$

$$\left(-i\frac{\partial}{\partial z} + \beta_n(\omega)\right)c_B^n(\omega, z) = -\frac{1}{2\beta_n(\omega)}F_n(c(\omega, z)), \qquad (17)$$

whereas the total spectrum is

$$c(\omega, z) = c_F(\omega, z) + c_B(\omega, z).$$

Note that if the backward spectrum is sufficiently small and it can be approximately neglected ($c_B \approx 0$), then $c \approx c_F$ and only Eq. (13) or (16) matters. It is also interesting to note that small values of the backward spectrum imply small nonlinearities, as can be checked by going to the $c_B \rightarrow 0$ limit in Eq. (14) or (17). In such a limit, nonlinearities dissapear independently of the nature of their origin: $F(c) \rightarrow 0$. We will return to this point in the last section of this paper.

IV. SPINOR REPRESENTATION OF THE FORWARD-BACKWARD EQUATIONS

It is possible to put the first-order FBE's in a more compact form by using a spinor representation for the forward and backward spectral functions, c_F and c_B. We start noticing that the FBE's (with discrete-frequency indices) can be rewritten as

$$\left(-i\frac{d}{dz} - \beta_{nj}\right)c_F^{nj} = \sum_{n'j'}[N_{nn'}^{jj'}(c)c_F^{n'j'} + N_{nn'}^{jj'}(c)c_B^{n'j'}],$$

$$\left(-i\frac{d}{dz} + \beta_{nj}\right)c_B^{nj} = \sum_{n'j'}[-N_{nn'}^{jj'}(c)c_F^{n'j'} - N_{nn'}^{jj'}(c)c_B^{n'j'}].$$

$$(18)$$

We have assumed that the nonlinear function F can be expressed as the action of a field-dependent operator $M(c)$ on the spectral function c—that is, $F_n^j(c) \equiv \sum_{n'j'}M_{nn'}^{jj'}(c)c_{n'}^{j'}$. This is the situation that applies to all type of nonlinearities that can be expanded in a power series. The simplest case is the aforementioned Kerr cubic nonlinearity, for which $M_{nn'}^{jj'}(c) = \sum_{mm'}^{k'}c_{m'}^{k'*}T_{nm'n'm}^{jk'j'(j+k'-j')}c_m^{j+k'-j'}$. Moreover, in conservative systems—that is, those for which the Hamiltonian is conserved and real—the M operator is forced to be self-adjoint. The N operator appearing in Eq. (18) is nothing but $N_{nn'}^{jj'}(c) \equiv 1/(2\beta_{nj})M_{nn'}^{jj'}(c)$.

The equations above can also be written in a matrix form

$$-i\frac{d}{dz}\begin{pmatrix} c_F^{nj} \\ c_B^{nj} \end{pmatrix} - \beta_{nj}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c_F^{nj} \\ c_B^{nj} \end{pmatrix}$$

$$= \sum_{n'j'} N_{nn'}^{jj'}(c)\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} c_F^{n'j'} \\ c_B^{n'j'} \end{pmatrix}. \qquad (19)$$

Then, introducing the bispinor ψ, defined as

$$\psi_n^j \equiv \begin{pmatrix} c_F^{nj} \\ c_B^{nj} \end{pmatrix}, \qquad (20)$$

and the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

noticing that

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \sigma_3 + i\sigma_2,$$

we see that Eq. (19) can be written in a bispinor form notation (a sum over repeated indices is assumed)

$$-i\frac{d}{dz}\psi_n^j = [(\beta_{n'j'}\delta_{nn'}\delta_{jj'} + N_{nn'}^{jj'})\sigma_3 + iN_{nn'}^{jj'}\sigma_2]\psi_{n'}^{j'}. \quad (21)$$

Certainly, N is a nontrivial operator in frequency and mode spaces. However, it is proportional to the identity operator when acting on the F-B internal degrees of freedom of the bispinor ψ. If we use continuum-frequency notation, Eq. (21) can be written as

$$-i\frac{\partial}{\partial z}\psi_n(\omega, z) = \beta_n(\omega)\sigma_3\psi_n(\omega, z) + \sum_{n'}\int d\omega' N_{nn'}(\omega, \omega'; \psi)$$

$$\times(\sigma_3 + i\sigma_2)\psi_{n'}(\omega', z). \qquad (22)$$

Despite its different appearence, the first-order spinor equation for the pulse spectrum contains exactly the same information on dynamics than the original second-order equation (8). The spinor representation of the FBE's has common features with the Dirac equation for a particle in $1+1$ dimensions. In $1+1$ dimensions, the original algebra of 16 4×4 matrices (Dirac matrices) of the four-dimensional (4D) Dirac equation is reduced to 4 2×2 matrices constituted by the identity and Pauli matrices. The FBE spinor (20), formed by the forward and backward spectral functions, plays the role of the Dirac spinor, constituted by the positive and negative-energy components of the particle wave function. First-order spinor equations involving forward-backward components have been also found in the description of different optical systems such as laser cavities [10,11].

V. CONSERVED QUANTITIES

The goal of this section is to find the conserved quantity associated to FBE's in the most general case. Our only assumption will be the conservative and lossless character of the system, which mathematically will be reflected in the fact that the M operator is self-adjoint ($M = M^\dagger$) and that the dis-

persion relation function β is real for all modes. We consider as a starting point the discrete-frequency FBE's in their spinor representation (21). Now we proceed to redefine the labeling of this equation following the same procedure we used to incorporate polarization indices into modal indices in Sec. III. Since frequency (j) and modal indices (n) are always paired together in Eq. (21), we can incorporate both in a new single index: $(n, j) \rightarrow p$, $(n', j') \rightarrow p'$. In this way, Eq. (21) becomes (a sum over repeated indices is assumed, as before)

$$-i\frac{d}{dz}\psi_p = [(\beta_{p'}\delta_{pp'} + N_{pp'})\sigma_3 + iN_{pp'}\sigma_2]\psi_{p'}. \qquad (23)$$

This form of the equation enables us to introduce the matrix operators Ψ, B, and N, whose elements are given by ψ_p, $\beta_{p'}\delta_{pp'}$, and $N_{pp'}$, respectively. Therefore, we can write ($\dot{\Psi} \equiv d\Psi/dz$)

$$-i\dot{\Psi} = \left\{\left(B + \frac{1}{2}B^{-1}M\right)\sigma_3 + i\frac{1}{2}B^{-1}M\sigma_2\right\}\Psi, \qquad (24)$$

where we have made use of the relation between the N and M operators: $N_{pp'} = 1/(2\beta_p)M_{pp'} \Rightarrow N = (1/2)B^{-1}M$. We will also need the adjoint of the above equation:

$$i\dot{\Psi}^\dagger = \Psi^\dagger\left\{\left(B + \frac{1}{2}MB^{-1}\right)\sigma_3 - i\frac{1}{2}MB^{-1}\sigma_2\right\}, \qquad (25)$$

where we have used the properties that $B = B^\dagger$ (since B is diagonal and β is real) and $M = M^\dagger$, together with the self-adjointness of the Pauli matrices. Note that both B and M are proportional to the identity matrix when they act on the F-B components of the bispinor Ψ, and for this reason they commute with Pauli matrices. Next, we left-multiply Eq. (24) by $\Psi^\dagger B\sigma_3$ and right-multiply Eq. (25) by $B\sigma_3\Psi$, to obtain ($\sigma_3^2 = 1$, $\sigma_2\sigma_3 = i\sigma_1$)

$$i\dot{\Psi}^\dagger B\sigma_3\Psi = \Psi^\dagger\left\{B^2 + \frac{1}{2}M + \frac{1}{2}M\sigma_1\right\}\Psi,$$

$$-i\Psi^\dagger B\sigma_3\dot{\Psi} = \Psi^\dagger\left\{B^2 + \frac{1}{2}M + \frac{1}{2}M\sigma_1\right\}\Psi.$$

We achieve the desired result by substracting the previous equations:

$$i(\dot{\Psi}^\dagger B\sigma_3\Psi + \Psi^\dagger B\sigma_3\dot{\Psi}) = 0 \Leftrightarrow \frac{d}{dz}(\Psi^\dagger B\sigma_3\Psi) = 0. \quad (26)$$

The conserved quantity is thus $Q = \Psi^\dagger B\sigma_3\Psi$ or, after reintroducing indices, $Q = \sum_{n,j}\psi_{nj}^\dagger\beta_{nj}\sigma_3\psi_{nj}$ (discrete frequency) or $Q = \sum_n\int d\omega\psi_n^\dagger(\omega)\beta_n(\omega)\sigma_3\psi_n(\omega)$ (continuous frequency). In terms of the original forward and backward components of the bispinor ψ, the conserved quantity has the following form:

$$Q = \sum_n\int d\omega\beta_n(\omega)[c_{Fn}^*(\omega, z)c_{Fn}(\omega, z) - c_{Bn}^*(\omega, z)c_{Bn}(\omega, z)].$$

$$(27)$$

The physical meaning of this conserved quantity can help us to understand its particular form. Equation (27) is the modal frequency version of the axial component of the Poynting vector, as one can check by recalling that, for the case of Maxwell's equations in the scalar approximation, $\mathcal{P}_z \sim \int_{\mathbb{R}^2} i\phi^* \partial_z \phi$. This represents the amount of electromagnetic energy traversing a section of the system per unit time. For this reason, since we are dealing in fact with the axial flux of the electromagnetic field, it is natural that the quantity Q, the total electromagnetic axial flux, can be considered as the sum of the positive forward axial flux $(+Q_F)$ and the negative backward axial flux $(-Q_B)$: $Q = Q_F - Q_B$, where $Q_F = \Sigma_n \int \beta_n c_{Fn}^* c_{Fn}$ and $Q_B = \Sigma_n \int \beta_n c_{Bn}^* c_{Bn}$.

VI. PARTICULAR CASES

An interesting case to consider is that occurring when one neglects all F-B interactions. In general, the FBE's (16) and (17) mix the F-B components due to the presence of the nonlinear function $F(c) = F(c_F + c_B)$ on the right-hand side of both equations. When F-B interactions can be neglected— i.e., when either c_F or c_B are very small—FBE's decouple and we obtain two separate equations for c_F and c_B. In such a case, forward and backward axial fluxes are conserved independently: $dQ_F/dz = 0$ and $dQ_B/dz = 0$. The demonstration of this property closely follows the general proof previously described for the total flux Q. Let us consider FBE's in their discrete-frequency form [Eqs. (13) and (14)] when all F-B mixing terms are neglected—that is, when $F(c) \approx F(c_F)$ in Eq. (13) because $c \approx c_F$ $(c_B \approx 0)$ or $F(c) \approx F(c_B)$ in Eq. (14) because $c \approx c_B$ $(c_F \approx 0)$. We will consider forward components only (backward analysis is completely analogous). By relabeling the frequency and modal indices together into a new index and by introducing the matrix notation in the same way we used before to obtain Eq. (24), the equation for forward components adopts the form

$$-i\dot{c}_F = \left(B + \frac{1}{2}B^{-1}M(c_F)\right)c_F. \tag{28}$$

In this case, the equation for the backward component would correspond to a very weak field $(c_B \approx 0)$ and, thus, it would be basically linear: $-i\dot{c}_B \approx -Bc_B$. For our purposes, we also need the adjoint equation of Eq. (28):

$$i\dot{c}_F^\dagger = c_F^\dagger\left(B + \frac{1}{2}M(c_F)B^{-1}\right).$$

Left-multiplying Eq. (28) by $c_F^\dagger B$ and right-multiplying the previous equation by Bc_F, one gets

$$ic_F^\dagger B\dot{c}_F = c_F^\dagger\left(B^2 + \frac{1}{2}M(c_F)\right)c_F,$$

$$-i\dot{c}_F^\dagger Bc_F = c_F^\dagger\left(B^2 + \frac{1}{2}M(c_F)\right)c_F.$$

The desired conservation law comes now after substracting the above equations:

$$\frac{d}{dz}(c_F^\dagger Bc_F) = c_F^\dagger B\dot{c}_F + \dot{c}_F^\dagger Bc_F = 0.$$

Certainly, $Q_F = c_F^\dagger Bc_F = \Sigma_n \int \beta_n c_{Fn}^* c_{Fn}$ is the conserved forward axial flux. Conservation of $Q_B = c_B^\dagger Bc_B = \Sigma_n \int \beta_n c_{Bn}^* c_{Bn}$ is a trivial issue because c_B fulfills the linear equation $-i\dot{c}_B \approx -Bc_B$. An analogous proof shows that Q_F and Q_B are also conserved when forward components are neglected instead.

An alternative analysis can be formulated in the light of conserved phase symmetries when F-B interactions are neglected. Both forward and backward axial fluxes can be also envisaged as the conserved charges associated to independent global U(1) symmetry transformations on F and B components, respectively. This is clear in Eq. (28), which is invariant under U(1)$_F$ global phase transformations

$$c_F \to e^{i\theta_F}c_F, \tag{29}$$

when $M(c_F) = M(e^{i\theta_F}c_F)$. This is the situation for all type of nonlinearities that can be expanded in odd power series in conservative lossless systems. Again, the simplest example is a cubic or Kerr nonlinearity, for which $M(c) = c^*Tc$. Since the equation for backward components is basically linear $(-i\dot{c}_B \approx Bc_B)$, global U(1)$_B$ invariance $(c_B \to e^{i\theta_B}c_B)$ is trivially fulfilled as well. U(1)$_F \otimes$ U(1)$_B$ invariance requires the independent conservation of its associated U(1)$_F$ and U(1)$_B$ charges, which are nothing but the F axial flux (Q_F) and the B axial flux $(-Q_B)$—that is, $dQ_F/dz = 0$ and $-dQ_B/dz = 0$. If the situation were the opposite and forward components were neglected, a complete analogous analysis for backward components would hold leading to the same conclusion.

When F-B interactions cannot be neglected, the previous argument does not hold and neither Q_F nor Q_B is conserved separately in conservative lossless systems. From a symmetry point of view, this can be understood by the fact that FBE's (16) and (17) are no longer invariant under independent F-B phase transformations $(c = c_F + c_B \to \tilde{c} = e^{i\theta_F}c_F + e^{i\theta_B}c_B)$ because in that case $M(\tilde{c}) = M(e^{i\theta_F}c_F + e^{i\theta_B}c_B) \neq M(c_F + c_B) = M(c)$, as can be easily checked for a cubic nonlinearity. Using the language of group theory, one would state that U(1)$_F \otimes$ U(1)$_B$ symmetry is broken. However, there is a residual phase symmetry remaining in FBE's (16) and (17) when U(1)$_F \otimes$ U(1)$_B$ invariance is broken by F-B interactions. FBE's are still invariant under simultaneous global phase tranformations on F and B components $(c_F \to e^{i\theta}c_F$ and $c_B \to e^{i\theta}c_B$, which implies $c \to e^{i\theta}c)$ provided that $M(e^{i\theta}c) = M(c)$. Note that this U(1) symmetry is a particular case of the higher-order U(1)$_F \otimes$ U(1)$_B$ symmetry when $\theta_F = \theta$ and $\theta_B = \theta$. Since U(1)$_F \otimes$ U(1)$_B$ symmetry is broken, the U(1)$_F$ charge—i.e., the F axial flux (Q_F)—and the U(1)$_B$ charge— i.e., the B axial flux $(-Q_B)$—are not conserved. However, in such a situation the remaining U(1) symmetry guarantees the conservation of a new quantity—namely, the *sum* of the U(1)$_F$ and U(1)$_B$ charges. That is, $Q_F + (-Q_B)$ has to be conserved. The total axial electromagnetic flux $Q = Q_F - Q_B$ appears then as the conserved charge of the U(1) symmetry associated with the breaking pattern U(1)$_F \otimes$ U(1)$_B$ $\to$ U(1)$_{F+B}$.

There are two different and interesting physical situations in which the analysis described above for systems in which F-B interactions can be neglected is valid. The first one is general forward pulse propagation, which naturally takes place in the frequency domain or, equivalently, in the time domain. The second one is monochromatic (or quasimonochromatic) nonparaxial forward evolution describing the nonlinear propagation of spatial structures. It is remarkable that despite the distinct nature of both phenomena, they are just particular cases of FBE's and, thus, equally described by the same formalism. We will study them separately in the next two subsections.

A. Forward pulse propagation

When one neglects F-B interactions—for example, by eliminating backward components—the most general form of FBE's is given by Eq. (28), which in continuum-frequency notation reads

$$-i\frac{\partial}{\partial z}c_F^n(\omega,z) = \beta_n(\omega)c_F^n(\omega,z) + \frac{1}{2\beta_n(\omega)}$$

$$\times \sum_{n'} \int d\omega' M_{nn'}(\omega,\omega';c_F)c_F^{n'}(\omega',z).$$

(30)

The nature of the nonlinearities defines the particular form of the nonlinear modal matrix function $M_{nn'}(\omega,\omega')$. According to what was explained in Sec. III, the functional form of the nonlinear function F and, thus, of the M matrix function can be systematically constructed out of the mode amplitudes of the linear propagation problem together with the standard nonlinear coefficients $(\chi^{(3)},\chi^{(5)},\dots)$. Thus, many formal properties of M will be inherited from linear modes. The dependence of M on modal indices and frequency will have much to do with the particular dependence of the linear mode amplitudes on spatial coordinates and frequency. Different physically well-grounded assumptions on the properties of M on modal (and polarization) indices and frequency can be then made by analyzing linear mode characteristics. The second element to take into account is the extension of the frequency spectrum $c_F(\omega)$. A considerable simplification is achieved for sufficiently narrow spectra, whereas wide bandwiths, such as those naturally appearing in highly nonlinear fibers (e.g., in supercontinuum generation), would demand one considers the frequency dependence of Eq. (30) in its total extent.

It is clear that, in the most general case, Eq. (30) involves an intricate dynamics since the nonlinear matrix function $M_{nn'}(\omega,\omega')$ leads to nonzero couplings between different frequency, modal and even polarization components (recall that polarization indices are included). Thus, even starting from a simple spectral, single-mode, single-polarization configuration, if the system had no physical mechanism to minimize the great variety of couplings induced by M, spectral evolution, as described by Eq. (30), would generate a more and more complicated spectral function by exciting new frequency, modal, and polarization components.

An important feature of Eq. (30) is the existence of an inherent spatial modal interference of nonlinear origin, represented by the nondiagonal nature of the nonlinear matrix function in the modal indices—$M_{nn'} \neq 0$ $(n \neq n')$—in the most general case. In some special situations, however, this modal interplay can be zero or it can be just neglected ($M_{nn'}=0$ or $M_{nn'} \approx 0$, when $n \neq n'$). Exact cancellation of matrix elements occurs by symmetry considerations in most cases. A simple example would be given by a rotationally invariant system. Its linear modes are solutions with well-defined angular momentum and, consequently, the modal index of the spectral function c_F^n is thus labeled by the angular momentum index l. In such a case, angular momentum conservation requires that no mixing of spectral components with different angular momentum occurs, thus eliminating nondiagonal terms in the nonlinear matrix function corresponding to different values of the angular momentum index l; that is, M has to be diagonal in these indices: $M_{nn'} \sim \delta_{ll'}$. In other cases, some nondiagonal matrix elements can be negligible because of the different shapes of linear modes amplitudes involved in the calculation of the overlapping integrals appearing in the definition of M (for example, for a Kerr cubic nonlinearity, $M_{nn'} \sim c_{m'}^* c_m \int_{R^2} \phi_m^* \phi_{n'}^* \phi_m \phi_n$). In some cases, these integrals can be very small for reasons that do not rely on the presence of particular symmetries.

Whatever these reasons may be, in the case that modal interplay does not exist or this can be neglected—that is, when $M_{nn'}=0$ or $M_{nn'} \approx 0$ if $n \neq n'$—then Eq. (30) decouples into independent equations for every mode index,

$$-i\frac{\partial}{\partial z}c_F^n(\omega,z) = \beta_n(\omega)c_F^n(\omega,z)$$

$$+ \frac{1}{2\beta_n(\omega)} \int d\omega' M_n(\omega,\omega';c_F)c_F^n(\omega',z),$$

(31)

which, in turns, leads to independent conservation laws for the different modal F and B axial fluxes: $dQ_F^{(n)}/dz=0$ and $dQ_B^{(n)}/dz=0$ ($Q_F^{(n)} \equiv \int_\omega \beta_n c_F^{(n)*} c_F^{(n)}$ and $Q_B^{(n)} \equiv \int_\omega \beta_n c_B^{(n)*} c_B^{(n)}$).

In nonlinear propagation in optical single-mode fibers it is commonly assumed that the spatial dependence of the electric propagating field is just given by the amplitude of the fundamental mode of the fiber. In our context, this statement is equivalent to say that the matrix M is one dimensional and involves the fundamental mode only. The equation describing forward nonlinear propagation in such a fiber would be Eq. (31) for $n=0$ (fundamental mode). Even if the fiber is not single mode but involves modes widely separated by large gaps in β's, as in some highly nonlinear microstructured fibers, the same equation can still remain approximately valid. The reason is that in such a case intermodal interactions are relatively suppressed with respect to modal self-interactions because of the very different forms of linear mode amplitudes, originated by both the discrete symmetry of the fiber and their very different values of β, leading to zero or small overlapping integrals and, thus, to small values of $M_{nn'}$ when $n \neq n'$. The resulting approximated equation— Eq. (31) restricted to the fundamental mode $(n=0)$—is

equivalent to the forward Maxwell equation for a highly non-linear microstructured fiber used to adequately describe supercontinuum generation in this type of fibers [3].

B. Nonparaxial spatial evolution

Monochromatic (or, in practice, quasimonochromatic) propagation is also a particular situation that can be described using FBE's. Equations (16) and (17) remain certainly valid when the frequency content of the F and B spectral functions c_F and c_B is restricted to a single frequency ω_0. In such a case, only intermodal interactions or modal self-interactions (including polarization) play a role since the nonlinear matrix function has a trivial dependence on frequency $M_{nn'}(\omega, \omega'; c) = M_{nn'}(\omega_0; c)$ fixed by the propagation frequency ω_0. Mathematically, FBE's become [in order to symplify the notation, $\beta_{0n} \equiv \beta_n(\omega_0)$, $M_{nn'}(c) \equiv M_{nn'}(\omega_0; c)$, $c(z) \equiv c(z; \omega_0)$]

$$-i\frac{\partial}{\partial z}c_F^n(z) = \beta_{0n}c_F^n(z) + \frac{1}{2\beta_{0n}}\sum_{n'} M_{nn'}(c)c_{n'}(z), \quad (32)$$

$$-i\frac{\partial}{\partial z}c_B^n(z) = -\beta_{0n}c_B^n(z) - \frac{1}{2\beta_{0n}}\sum_{n'} M_{nn'}(c)c_{n'}(z), \quad (33)$$

and $c = c_F + c_B$.

Since the previous FBE's involve spatial modal indices only, it is interesting to write them also in the spatial domain, so that spatial degrees of freedom appear explicitly. This process is the inverse of the one we followed to obtain the second-order modal equation in Sec. II; that is, we reintroduce the spatial field amplitudes as $\phi_{F,B}(x,y,z) = \sum_n c_{F,B}^n(z)\phi_n(x,y)$, ϕ_n being the eigenfunctions of the linear operator $L_0 \equiv \nabla_t^2 + k_0^2 n_0^2$. β_{0n} are the eigenvalues of the L_0 operator and $n_0 \equiv n_{\omega_0}$ is the inhomogeneous refractive index, both evaluated at the fixed frequency $\omega_0 = k_0/c$, respectively. The outcome is

$$\left(-i\frac{\partial}{\partial z} - L_0^{1/2}\right)\phi_F = \frac{1}{2}L_0^{-1/2}M(\phi)\phi, \quad (34)$$

$$\left(-i\frac{\partial}{\partial z} + L_0^{1/2}\right)\phi_B = -\frac{1}{2}L_0^{-1/2}M(\phi)\phi, \quad (35)$$

together with $\phi = \phi_F + \phi_B$. The nonlinear term $M(\phi)\phi$ includes all types of nonlinearities in the spatial domain that permit an expansion in power series. The usual example is the Kerr nonlinearity $M(\phi)\phi \sim (\phi^*\phi)\phi$.

It is also an interesting exercise to prove that one can derive the standard second-order wave equation from the first-order spatial FBE's (34) and (35). If we sum and substract Eqs. (34) and (35), we obtain

$$-i\frac{\partial}{\partial z}(\phi_F + \phi_B) - L_0^{1/2}(\phi_F - \phi_B) = 0 \quad (36)$$

and

$$-i\frac{\partial}{\partial z}(\phi_F - \phi_B) - L_0^{1/2}(\phi_F + \phi_B) = L_0^{-1/2}M(\phi)\phi, \quad (37)$$

respectively. By applying $(i\partial/\partial z)$ to Eq. (36), susbtituting Eq. (37) into the resulting equation, and taking into account that $\phi = \phi_F + \phi_B$, one gets the Helmholtz-type nonlinear wave equation

$$\left(\frac{\partial^2}{\partial z^2} + \nabla_t^2 + k_0^2 n_0^2 + M(\phi)\right)\phi = 0.$$

As mentioned in Sec. III and is also evident in Eq. (35), the complete elimination of backward components ($\phi_B \to 0$) implies the vanishing of nonlinearities ($M \to 0$). Conversely, if $M \neq 0$ then backward components are necessarily generated: $\phi_B \neq 0$ (even if they did not initially exist). Therefore, there exists a close link between backward amplitudes and the nonlinear function M. Certainly, in most experimental situations in which an axially invariant system is axially illuminated along a privileged (say, forward) direction exclusively, backward components are small as compared to forward amplitudes. However, as backward FBE's show, they cannot be identically zero in the presence of nonlinearities. They are exactly zero only in the case the system behaves linearly. In such a case, since the system is axially invariant and is illuminated in the forward direction only, there would be no axial inhomogeneities that could produce linear reflections. Thus, small backward amplitudes are expected to be generated by small nonlinearities so that $\phi_B \to 0$ when $M \to 0$ (linear case). Our interest lies now in the quantification of the relation between ϕ_B and M in the small backward amplitud regime ($\phi_B = \delta\phi_B \ll 1$).

In order to clarify the calculation, we parametrize the "size" of the nonlinearity by redefining the nonlinear function as $M = \gamma\bar{M}$, where γ is a dimensionless real parameter. In this way, we can approach the linear case ($M \to 0$) by taking the limit $\gamma \to 0$. It is easy to realize that γ has to be proportional to the input power $\gamma \sim P$ when γ is small. Certainly, M is a function of the input power verifying that $M \to 0$ as $P \to 0$. Assuming, as usual, that this dependence is analytical this implies that $M \sim P$ when $P \to 0$. The same argument for γ leads to $M \sim \gamma$ for small values of γ. Therefore, the small-γ and small-P regimes are, in fact, the same. From a physical point of view, this provides a physical meaning to the dimensionless parameter γ in the small nonlinearity regime: $\gamma \sim P$. Note that, in the general case of a nonlinearity that can be expanded in power series in ϕ, the auxiliar nonlinear function $\bar{M}$ can depend itself on γ. However, for our purposes, it is enough to consider that $M = O(\gamma)$ [and thus $\bar{M} = O(1)$] because we are going to be interested in leading-order terms.

Following the reasoning above, we consider the total field amplitude as the sum of a large forward amplitude and a small backward amplitude: $\phi = \phi_F + \delta\phi_B$. Besides, the small backward amplitude is a function of the nonlinear parameter γ—$\delta\phi_B = \delta\phi_B(\gamma)$—and it has to verify that $\delta\phi_B(0) = 0$, since we are assuming that no backward radiation is present in the absence of nonlinearities. The nonlinear function can be then

expanded as $M(\phi_F + \delta\phi_B) = M(\phi_F) + (\partial M / \partial\phi|_{\phi_F})\delta\phi_B + O(\delta\phi_B)^2$. Substituing the total amplitude $\phi = \phi_F + \delta\phi_B$ and the expansion of M in Eq. (35) and keeping the leading-order terms in $\delta\phi_B$ only, one gets

$$\left[-i\frac{\partial}{\partial z} + L_0^{1/2} + \frac{1}{2}L_0^{-1/2}\left(M(\phi_F) + \phi_F\frac{\partial M}{\partial\phi_F} \right) \right]\delta\phi_B$$

$$= -\frac{1}{2}L_0^{-1/2}M(\phi_F)\phi_F + O(\delta\phi_B)^2.$$

Since $M = O(\gamma)$, we can find the leading-order term in γ for $\delta\phi_B$ from the previous equation:

$$\delta\phi_B = -\frac{1}{2}\left(-i\frac{\partial}{\partial z} + L_0^{1/2} \right)^{-1} L_0^{-1/2}M(\phi_F)\phi_F + O(\gamma)^2,$$

so that $\delta\phi_B$ is also $O(\gamma)$. We can proceed analogously with the forward FBE (34) to find

$$\left(-i\frac{\partial}{\partial z} - L_0^{1/2} \right)\phi_F = \frac{1}{2}L_0^{-1/2}M(\phi_F)\phi_F + \frac{1}{2}L_0^{-1/2}$$

$$\times\left(M(\phi_F) + \phi_F\frac{\partial M}{\partial\phi_F} \right)\delta\phi_B + O(\delta\phi_B)^2,$$

which, after introducing γ dependences [$M = O(\gamma)$, $\delta\phi_B = O(\gamma)$], becomes

$$\left(-i\frac{\partial}{\partial z} - L_0^{1/2} \right)\phi_F = \frac{1}{2}L_0^{-1/2}M(\phi_F)\phi_F + O(\gamma)^2. \quad (38)$$

Therefore, neglecting backward components is equivalent to consider FBE's up to $O(\gamma)^2$ terms. Or, equivalently, the pure forward equation (that is, FBE's with $\phi \approx \phi_F$ and $\phi_B \approx 0$) is just a weak-nonlinearity approximation. More specificaly, it is the leading-order contribution in the nonlinear parameter γ to FBE's. For this reason, it is possible to find equivalent forms to the forward equation different than Eq. (38). An interesting alternative version of the forward equation is easily obtained by using the property

$$L_0^{1/2}\phi_F + \frac{1}{2}L_0^{-1/2}M(\phi_F)\phi_F + O(\gamma)^2$$

$$= L_0^{1/2}\left(1 + \frac{1}{2}L_0^{-1}M(\phi_F) \right)\phi_F + O(\gamma)^2$$

$$= [L_0 + M(\phi_F)]^{1/2}\phi_F + O(\gamma)^2,$$

which allows us to write Eq. (38) as

$$-i\frac{\partial}{\partial z}\phi_F = [L_0 + M(\phi_F)]^{1/2}\phi_F + O(\gamma)^2. \quad (39)$$

The previous version of the forward equation has been used to simulate monochromatic nonlinear propagation of spatial structures in photonic crystal fibers [12] in the nonparaxial regime. Despite the pure forward equation being first order in z, it has an intrinsic nonparaxial nature. Unlike in the nonlinear Schrödinger equation, the standard integral $\int \phi_F^*\phi_F$ is not a conserved quantity. This can be clearly seen if one writes the evolution operator associated with Eq. (39) for an infinitesimal axial step ϵ:

$$\phi_F(z + \epsilon) = \exp i\epsilon[L_0 + M(\phi_F(z))]^{1/2}\phi_F(z).$$

This evolution operator is not unitary because the operator $L_0 + M(\phi_F(z))$, despite it being self-adjoint, it is not positive definite, inasmuch as it can have negative eigenvalues corresponding to evanescent waves ($\beta^2 < 0$). The loss of unitarity is due to this evanescent modes leading to the nonconservation of the integral $\int \phi_F^*\phi_F$.

VII. CONCLUSIONS

The experimental availability of high nonlinearities is expected to unveil a number of new effects that will force us to extract information from Maxwell's equations in a more accurate manner. In this paper we shed some light on the problem of revealing the close interplay between backward components and nonlinearities in axially invariant systems. With a minimum amount of approximations, we have been able to find a system of two coupled first-order equations for the forward and backward spectral components of the electromagnetic field, the so-called forward-backward equations. The explicit appearence of forward and backward components as well as of their nonlinear couplings in these equations is useful to quantify under which conditions nonlinearly generated backward components can be relevant in a new scenario of highly nonlinear effects.

From the formal point of view, the FBE's are especially appealing in the sense that they admit a simple bispinor representation that closely resembles that of a Dirac equation for the positive and negative components of a fermion wave function in 1+1 dimensions. Similarly to the Dirac equation, the use of the algebraic properties of the spinor FBE's allows us to obtain the conserved quantities associated with them in an elegant way. In the same manner, all conserved quantities also admit an interpretation as conserved charges associated with phase symmetries.

The dimensional reduction is a remarkable issue of the modal approach followed here. The original 3+1 dimensions (3 spatial, 1 frequency) of the starting wave equation for $\mathbf{E}_\omega(x, y, z)$ [Eq. (1)] are reduced to 1+1 (1 spatial, 1 frequency) in the FBE's. The modal approach is a way of "integrating out" the transverse spatial degrees of freedom (x and y). Of course, the coupling between tranverse degrees of freedom in Eq. (1) does not dissapear in FBE's. It transforms into the couplings between the different modal components of the spectral function c_n which are mathematically encoded in the nonlinear matrix function $M_{nn'}$. For the case in which only a few modal components are relevant, the process of dimensional reduction provides a dramatical simplification. Typical envelope equations for propagating pulses are the result of a similar process in which the propagation of only one linear mode (usually, the fundamental mode) is assumed. The FBE's, however, provide a natural way of dealing with a more complex modal structure and, moreover, they allow one to directly work with the frequency content of the propagating field—that is, with its spectral components $c_n(\omega, z)$. In this sense, there is no need to resort to the concept of pulse envelope (unless one wants to make contact to other approaches). For the same reason, the FBE's are equally suit-

able to describe pulse propagation with extremely large bandwiths since no assumption on the form of $c_n(\omega, z)$ is required. Ideally, they could handle any type of temporal-spectral behavior provided the only assumption needed, the "scalar approximation" (in its strong or weak form—see Sec. II), is reasonably fulfilled.

Another scenario in which these equations can be useful is that in which there are relevant spatio-temporal effects. The FBE's inherently include couplings between frequency and modal indices through the nonlinear matrix function $M_{nn'}(\omega, \omega'; c)$. Since modal indices correspond to spatial amplitudes of linear modes, the nonlinear matrix function $M_{nn'}(\omega, \omega'; c)$ is constructed out of these amplitudes as an overlapping integral (Sec. II). Thus, part of the frequency dependence of $M_{nn'}$ is due to the explicit dependence of linear mode amplitudes on frequency. In the ultrawide spectrum regime, this dependence cannot be neglected and, therefore, the contribution of several spatial modes can produce simultaneous couplings between modal indices and frequencies which can be naturally treated in the framework of the FBE's. In terms of the original 3+1 Maxwell's equations,

these effects would correspond to spatio-temporal phenomena, in which the spatial and temporal degrees of freedom of the electric field $\mathbf{E}(x, y, x, t)$ could not be factorized.

Sumarizing, the FBE's provide a general framework to deal with nonlinearities in axially invariant inhomogenous dielectric media, limited only to a reasonable validity of the "scalar approximation" (in its strong or weak form). As we have seen in Sec. VI, its generality can be made evident after analizing two apparently disconnected cases: forward pulse propagation (a purely temporal phenomenon) and monochromatic nonparaxial evolution of spatial structures (a purely spatial phenomenon). Both are equally and naturally described within the FBE formalism. The fast progress in nonlinear optics experiments provides effects of increasing complexity both in the spatial and time domains as well as in the interplay between forward and backward components. Strong correlations between spatial and time domains and forward and backward components will play a more and more important role. In this context, the FBE's can be a suitable and convenient tool to encompass a variety of different phenomena within a common framework.

[1] G. P. Agrawal, *Optics and Photonics*, 2nd ed. (Academic Press, San Diego, CA, 1995).

[2] J. K. Ranka, R. S. Windeler, and A. J. Stentz, Opt. Lett. **25**, 25 (2000).

[3] A. V. Husakou and J. Herrmann, Phys. Rev. Lett. **87**, 203901 (2001).

[4] S. Coen, A. H. L. Chau, R. Leonhardt, J. D. Harvey, J. C. Knight, W. J. Wadsworth, and P. S. J. Russell, Opt. Lett. **26**, 1356 (2001).

[5] K. J. Blow and D. Wood, IEEE J. Quantum Electron. **25**, 2665 (1989).

[6] P. V. Mamyshev and S. Chernikov, Opt. Lett. **15**, 1076 (1990).

[7] T. Brabec and F. Krausz, Phys. Rev. Lett. **78**, 3282 (1997).

[8] M. Kolesik, J. V. Moloney, and M. Mlejnek, Phys. Rev. Lett. **89**, 283902 (2002).

[9] Y. R. Shen, *The Principles of Nonlinear Optics* (Wiley, New York, 1984).

[10] H. G. Solari and R. Gilmore, Opt. Commun. **71**, 85 (1989).

[11] H. G. Solari and R. Gilmore, J. Opt. Soc. Am. B **7**, 828 (1990).

[12] A. Ferrando, M. Zacarés, P. F. de Córdoba, D. Binosi, and J. A. Monsoriu, Opt. Express **11**, 452 (2003).